**­­­A New Model of Motion (Liao et al. 2023)**

**Abstract:** Human motor behavior is an active interdisciplinary research area. One major topic of investigation is how human central nervous system (CNS) plans and controls motion.The minimum-jerk model minimizes the L-2 norm of the third derivative (the jerk) of the displacement. The resulting velocity curves are 5th order polynomials and are symmetric with one peak. To include asymmetric curves with multiple peaks, we propose a new model which is based on minimizing the sum of the L-2 norms of the jerk and the fractional derivative of order 1.5.

**Key words.** Motor behavior, Minimal jerk, Fractional calculus, Asymmetric with multiple peaks

**1. Introduction**

**Human motor behavior** is an active interdisciplinary research area. One major topic of investigation is how human central nervous system (CNS) plans and controls motion. In [7] (Wolpert and Ghahramani 2000), following principles regarding motion control models (with physiological evidence) are summarized:

1. Internal models (in three stages: inverse model, forward dynamics, and forward sensory) are fundamental for state estimation, prediction, context estimation, control, and learning.
2. Many models are based on optimizing a cost functional.
3. Statistical methods are needed in modeling noise and uncertainties of the state, the context, and the motor commands. Bayesian approach provides a tool for optimal estimation in the presence of uncertainty.

We will examine a popular model of normal human motor behavior and extend this model to include some effects of nonlocality, memory, delay, and uncertainty.

**2. The Minimum-Jerk Model**

The minimum-jerk model [2, 5, 6] is a smoothness principle, which assumes that the CNS selects the optimal trajectory (the displacement versus time) out of infinitely many possible ones in going from one point to another in a given time. The model is based on the minimization of the integral of square of the third time-derivative (the jerk) of the displacement. In one dimension, i.e. when the motion is restricted on the -axis, the cost functional for a displacement  is defined by

, (1)

subject to the following boundary conditions (with the normalized time 1 second):

, cm (2)

This model generates a movement with velocity profile in one symmetric arch, like the one shown in Figure 1 below. The minimal jerk model is characterized by 1.875, where the ratio of peak speed to average speed. The minimum-acceleration model is characterized by . For the minimum-snap model,  2.186. Their velocity profiles are also asymmetric with one peak.

We begin by examining the Euler-Lagrange differential equation which the minimizer of the cost function in (1) must satisfy. This equation is obtained by replacing by  in (1) then taking derivative with respect to , and let approach 0. Here is any smooth function defined on [0, 1] with

We get To get the last term, we applied the integration by parts three times, so that the three differentiation on are moved to , resulting in the sixth derivative

If is the minimizer of the functional (1), then . Sinceis arbitrary smooth function on [0, 1] with we get This is the Euler-Lagrange equation for the minimal jerk model. Its solutions are polynomials of order up to 5:

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With the required boundary conditions in (2), the velocity is a symmetric arch.

It is observed that if or , are added to the cost functional in (1), then we will get or , respectively. This means the Euler-Lagrange equation will have additional terms  or, which are even ordered derivatives, and the corresponding solutions are still restrictive.

To find a more flexible model, its Euler-Lagrange equation must have solutions beyond the powers of . Since the integration by parts always produce derivatives of  of even orders in the Euler-Lagrange equation if only integer derivatives of are used in the cost functional, we are led to using fractional derivatives in the new cost functional.

**3. A New Model for Motion**

We propose a new cost functional which aims to model asymmetric velocity profiles with single or multiple peaks. Our model is the sum of the cost functional in (1) and a functional involving a fractional derivative of the displacement. We demonstrate that the new model can produce velocity profiles of a symmetric arch with a wide range of , including and 1.618, as well as profiles with multiple peaks.

*t*, (3)

where is the fractional derivative of order 1.5.

Fractional calculus is useful for modeling nonlocal phenomena that involve memory, delay, and uncertainty [1, 3, 4]. An important fact is that fractional derivatives can be approximated as linear combinations of the first N integer derivatives, the function itself, and a constant vector [3] (Farid 2004). We convolute and its derivatives with the Gaussian filter to get the regularized displacement, velocity, acceleration, and jerk, again denoted as . We also convolute any variation and its derivatives with another low pass filter such as the sinc filter. Then, we have

, and where and are constants. We have and

We get

. Expanding the product of the integrand and performing integration by parts, we get. Including the first term in the new cost function, we get.

The Euler-Lagrange equation of the new cost functional is

Inspired by this differential equation, we propose to model the displacement of the motion by the general sixth order differential equation

, where are real numbers.

In terms of the velocity , the model equation becomes

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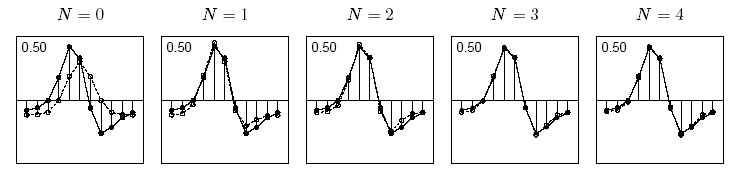
Let  where is a complex number. Then We have , and satisfies the following 5th order algebraic equation .

We get five roots: From the , we get a real solution  and four complex solutions, which give rise to five real solutions:

The model for is , (4)

which contains nine parameters

We approximate fractional derivatives by the first N integer derivatives, the function itself, and a constant vector as described in [3] (Farid 2004). The figures below from [3] show that the approximation to  of the Gaussian kernel with N = 3 is already particularly good by comparing the solid line, representing of the Gaussian kernel and the dotted line calculated by a linear combination of N terms.



In Examples 1 – 5, we demonstrate that the model approximates one arch symmetric motion very well. In Examples 6, 7, 8, we demonstrate asymmetric curves with one, two, and three peaks, respectively. In Example 9, we describe a method to fit the new model to a set of data points.

**4. Symmetric Curves with One Peak**

According to [5] (Richardson and Flash 2002), , the ratio of peak speed to average speed, is 1.5 for the minimum-acceleration model. For the minimum-jerk model, 1.875; for the minimum-snap model, 2.186. Psychophysical experiments reveal that our normal reaching movements have a ratio that is about 1.75. We will show by examples below, that curves with these values of can be achieved by our new model. In all the examples below, we use formula (4) with different values for c, w, p, q, and . In the Figures below, the horizontal axis represents time, the vertical axis represents , the velocity (speed). The time interval is [0, 1].

**Example 1.** 1.5 is achieved in our model with . The calculated peak() = 30. The average = 20. Thus, peak/average = 1.5.

**Example** **2.** 1.7575 is achieved with . The calculated peak = 35.15; average = 20; Thus peak/average 1.7575.

**Example 3.**1.875 is achieved by our model with, w = q = 2. The calculated peak() = 37.5; average () = 20. Thus, peak/ average = 1.875.

**Example 4.**  2.0 is achieved by our model with . The calculated peak() = 40.0.

Next example shows a one-arch velocity profile with 1/ = the Golden Section .

**Example 5.** he Golden Section (see Figure 1)

The reciprocal of the Golden Section is approximately 1.618, which is about the average of 1.5 (see example 1) and 1.7575 (see example 4). We are curious to know if 1.618 could be realized by the model. The answer is positive. Indeed, use  5.844588887, 2, we get peak() 32.360679774661520, the average() = 20. Thus, r = peak()/average() = 1.618033988733076, and 1/ average()/peak() 6.180339887563191E-001, which is in agreement with the Golden Section ≈ 6.180339887498949E-001 up to ten digits.

These examples confirmed that symmetric (one-arch) velocity profiles with a wide range of values for r can be realized from this model. Next three examples exhibit the potential capacity of the proposed model to describe more complicated curves.

**5. Asymmetric Curves with One, Two, and Three peaks**

**Example 6.** Asymmetric velocity profile (see Figure 2)

This example is done with.

**Example 7.** Velocity profile with two peaks (see Figure 3)

This example is done with.

**Example 8.** Velocity profile with three peaks (see Figure 4)

This example is done with.

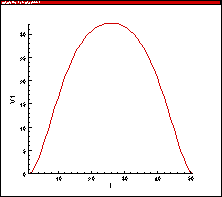
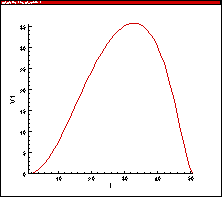
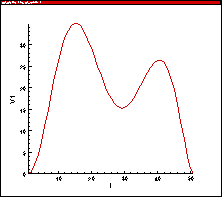
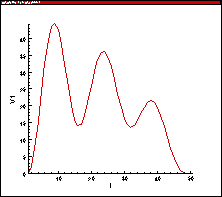
   

Figure 1. = 1.618 Figure 2. Asymmetric Figure 3. Two peaks Figure 4. Three picks

**Example 9.** Fitting the velocity model to a set of points

In this example, we fit the velocity model in (4) to a set of data points by determining the parameters , . The data set is in the () - coordinate format, as shown below

() = (3, 0), () = (3.25, 0.25), () = (4, 1.75), () = (4.5, 6), () = (5, 8), () = (5.5, 10.5), () = (6, 9.5), () = (6.75, 7), () = (7, 6.25), () = (8, 7.5), () = (9, 3.5), () = (9.5, 1), () = (11, 0), () = (12, 0.5), () = (13, 0.25), () = (14, 0).

Recall, that the basis functions are

and the model for the velocity is . (4)

Evaluate this model at , we get, for

Instead of imposing exactly , we use the least squares method to determine the nine parameters

Here we have renamed as respectively.

Define .

We minimize with respect to

At the minimum, the partial derivatives , and are equal to zero.

We have , where =

Thus, we get for (5)

We also have ,

where = and

=

Similarly, we have=, and

=.

We then get

, (6)

(7)

(8)

. (9)z

From the nine equations from (5) to (9), we solve for

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